

Equational reasoning with context-free families of string diagrams ^{*}

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Abstract. String diagrams provide an intuitive language for expressing networks of interacting processes graphically. A discrete representation of string diagrams, called string graphs, allows for mechanised equational reasoning by double-pushout rewriting. However, one often wishes to express not just single equations, but entire families of equations between diagrams of arbitrary size. To do this we define a class of context-free grammars, called B-ESG grammars, that are suitable for defining entire families of string graphs, and crucially, of string graph rewrite rules. We show that the language-membership and match-enumeration problems are decidable for these grammars, and hence that there is an algorithm for rewriting string graphs according to B-ESG rewrite patterns. We also show that it is possible to reason at the level of grammars by providing a simple method for transforming a grammar by string graph rewriting, and showing admissibility of the induced B-ESG rewrite pattern.

1 Introduction

A string diagram (Fig. 1(a)) consists of a collection of boxes (typically used to represent certain maps, processes, machines, etc.) connected by wires.

They are essentially labelled directed graphs, but with one important difference: wires, unlike edges, can be left open at one or both ends to form inputs and outputs, so they have an inherently compositional nature. Joyal and Street showed in 1991 that compositions of morphisms in any symmetric monoidal category can be represented using string diagrams [12], and recently there has been much interest in applying string diagram-based techniques in a wide variety of fields. In models of concurrency, they give an elegant presentation of Petri nets with boundary [20], in computational linguistics, they are used to compute compositional semantics for sentences [6], and in control theory, they represent signal-flow diagrams [3,4]. Equational reasoning for string diagrams has been used extensively in the program of categorical quantum mechanics [1], which provides elegant solutions to problems in quantum computation, information, and foundations using purely diagrammatic methods [5,11,8,9].

^{*} The final publication is available at Springer via http://dx.doi.org/10.1007/978-3-319-21145-9_9

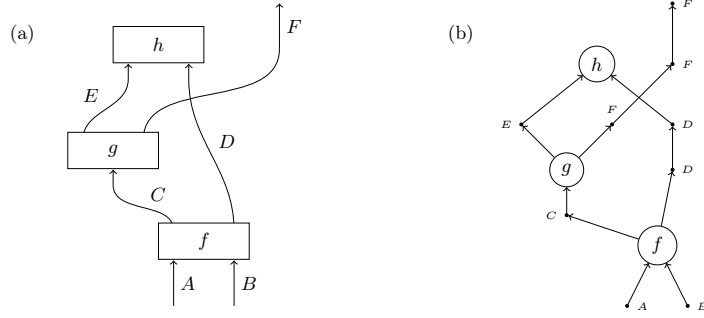


Fig. 1: A string diagram and its encoding as a string graph

All of these applications make heavy use of proofs by diagram rewriting. These are proofs whereby some fixed set of string diagram equations are used to derive new equations by cutting out the LHS and gluing in the RHS. For example, the following string diagram equation:

(1.1)

can be applied to rewrite a larger diagram as follows:

(1.2)

There are two things to note here. First, the LHS and RHS of string diagram equations always share a common boundary. In other words, we could think of the boundary as an invariant sub-diagram that embeds into the LHS and the RHS of the rule. Secondly, it is this invariant sub-diagram that is used to glue in the RHS once the LHS is removed. We can formalise this process using double-pushout (DPO) rewriting.

We begin by representing (directed) string diagrams as certain labelled, (directed) graphs called *string graphs* (see Fig. 1). Wires are replaced by chains of edges containing special dummy vertices called *wire-vertices*. By contrast, the ‘real’ vertices, labelled here f, g and h are called *node-vertices*. String graphs—originally introduced under the name ‘open graphs’ in [10]—have the advantage that they are purely combinatoric objects, as opposed to the geometric objects like string diagrams. As such, they form a suitable category for performing DPO rewriting. The DPO diagram associated with the rewrite (1.2) is given in Figure 2, where the top row is the string graph rewrite rule for (1.1), the left square is

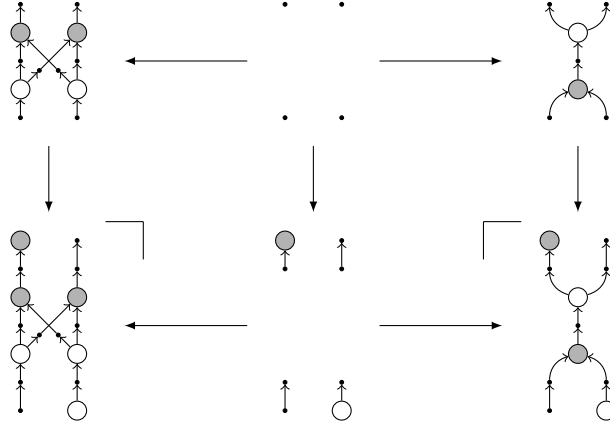


Fig. 2: DPO example

the pushout complement removing the LHS, and the right square is the pushout gluing in the RHS.

This technique has been used to mechanise proofs involving string diagrams, and forms the foundation of the diagrammatic proof assistant Quantomatic [15]. However, proving equations between single string diagrams is just half the story. Typically, one wishes to prove properties about entire families of diagrams. For example, suppose we have a node that serves as an n -fold ‘copy’ process. Then, one might require, as in Fig. 3(a), that connecting another node to the ‘copy’ node would result in n copies. Thus, we have an infinite family of (very similar) equations, one for each n .

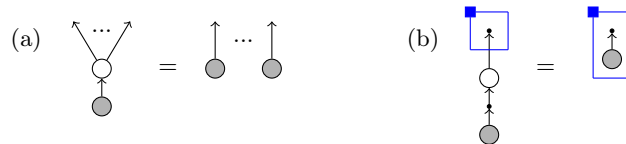


Fig. 3: An n -fold copy rule, and its formalisation using !-box notation.

Such a family of equations can be easily captured using a graphical syntax called *!-box notation*. Here, we can indicate that a subgraph (along with its adjacent edges) on the LHS and RHS of a rule can be repeated any number of times by wrapping a box around it, as in Fig. 3(b). This rewrite pattern can then be instantiated to a concrete rewrite rule by fixing the number of copies of each !-box to retain. A formal description of this instantiation process is given in [13]. This procedure is straightforward to mechanise, and is the main mechanism Quantomatic uses for reasoning about string diagram families. However, the types of string graph languages representable using !-box notation is quite lim-

ited. For example, the languages produced by string graphs with !-boxes always have finitely-bounded diameter and chromatic number.¹ Thus many naturally-occurring languages containing chains or cliques of arbitrary size are not possible.

In [14], we showed that the languages generated by string graphs with (non-overlapping) !-boxes can always be generated using a context-free vertex replacement grammar. Thus, we conjectured that a general notion of a context-free grammar for string graphs and string graph rewrite rules could produce a more expressive language for reasoning about families of string diagrams. In this paper, we will show that this is indeed the case.

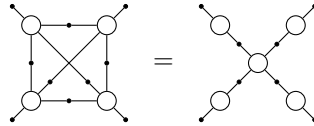
We begin by defining a context-free grammar for string graphs which is built on the well-known B-edNCE class of grammars. We call these grammars *boundary encoded string graph* (B-ESG) grammars, where ‘boundary’ here means the grammar satisfies the same boundary condition as B-edNCE grammars. An encoded string graph is a slight generalization of a string graph, where certain fixed subgraphs can be encoded as a single edge with a special label. A B-ESG grammar consists of a B-edNCE grammar for producing encoded string graphs, and a set of decoding rules for replacing these special edges. For example, this grammar:

$$\begin{array}{c}
 \text{S:} \boxed{\begin{array}{c} \bullet \\ \downarrow \\ \circ \\ \downarrow \\ \boxed{X} \end{array}} \quad \text{X:} \boxed{\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}} \quad \text{X:} \boxed{\begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \bullet \end{array}}
 \end{array}
 \quad
 \begin{array}{c}
 \circ \xrightarrow{\alpha} \circ \implies \circ \bullet \circ
 \end{array}
 \quad (1.3)$$

produces a complete graph of $n \geq 2$ white node-vertices, connected by wires, which would not be possible in the !-box language. We define B-ESG rules analogously as two grammars with identical non-terminals, and a 1-to-1 correspondence between their productions. Consider, for example, the following rule:

$$\begin{array}{c}
 \text{S:} \boxed{\begin{array}{c} \bullet \\ \downarrow \\ \circ \\ \downarrow \\ \boxed{X} \end{array}} \quad \text{X:} \boxed{\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}} \quad \text{X:} \boxed{\begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \bullet \end{array}}
 \end{array}
 =
 \begin{array}{c}
 \text{S:} \boxed{\begin{array}{c} \bullet \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \bullet \end{array}} \quad \text{X:} \boxed{\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}} \quad \text{X:} \boxed{\begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \bullet \end{array}}
 \end{array}
 \quad (1.4)$$

It will rewrite any complete graph of white node-vertices into a star with white node-vertices on every outgoing wire:



¹ For colourability and cliques in string diagrams, we treat chains of wire-vertices as single edges.

In this paper, we will define a family of context-free grammars suitable for equational reasoning with string diagrams. After giving formal definitions for string graphs in Section 2, we will define encoded string diagrams and B-ESG grammars in Section 3, as well as prove that a B-ESG grammar always yields a language consisting of well-formed string graphs. We also show that the B-ESG membership problem is decidable as well as the match enumeration problem. In Section 4 we will extend to B-ESG rewrite rules and show that these always form a language of well-formed string graph rewrite rules (i.e. the generated rules always have corresponding inputs/outputs). In Section 5, we give a simple example of meta-level reasoning with B-ESG grammars, whereby string graph rewrite rules can be lifted to admissible transformations of B-ESG grammars, thus proving entire families of diagram equations simultaneously. We then generalise this result to show how basic B-ESG on B-ESG rewriting is possible. In the conclusion, we discuss how these principles might be extended to more powerful versions of B-ESG on B-ESG rewriting and structural induction.

2 Preliminaries

Definition 1 (Graph [17]). A *graph* over an alphabet of vertex labels Σ and an alphabet of edge labels Γ is a tuple $H = (V, E, \lambda)$, where V is a finite set of vertices, $E \subseteq \{(v, \gamma, w) | v, w \in V, v \neq w, \gamma \in \Gamma\}$ is the set of edges and $\lambda : V \rightarrow \Sigma$ is the node labelling function. The set of all graphs with labels Σ, Γ is denoted $GR_{\Sigma, \Gamma}$.

Note that this notion of graphs forbids self-loops. This is a standard (and convenient) assumption in the vertex-replacement grammar literature. It will not get in our way, since we always use chains of wire-vertices to encode self-loops in string diagrams. Also, this notion of graph allows parallel edges only if they have different types. Again, this isn't problematic for our use case, because string graphs cannot have parallel edges (but they do allow parallel *wires*, cf. Definition 7).

2.1 B-edNCE grammars

We will focus on neighbourhood-controlled embedding (NCE) grammars. This is a type of graph grammar where non-terminal vertices are replaced by graphs according to a set of *productions*, each endowed with a set of *connection instructions* which determine how the new graph should be connected to the neighbourhood of the non-terminal. edNCE grammars are NCE grammars, with **e**dge labels and **d**irections, and B-edNCE grammars additionally impose the ‘boundary’ condition [18] which guarantees confluence for applications of productions. They form an important subclass of all confluent edNCE grammars (C-edNCE grammars) that is particularly easy to characterise.

Definition 2 (Graph with embedding [17]). A *graph with embedding* over labels Σ, Γ is a pair (H, C) , where H is a graph over Σ, Γ and $C \subseteq \Sigma \times \Gamma \times \Gamma \times V_H \times \{in, out\}$. C is called a *connection relation* and its elements are called *connection instructions*. The set of all graphs with embedding over Σ, Γ is denoted by $GRE_{\Sigma, \Gamma}$.

Graph grammars operate by substituting a graph (with embedding) for a non-terminal vertex of another graph. Connection instructions are used to introduce edges connected to the new graph based on edges connected to the non-terminal. A connection instruction $(\sigma, \beta/\gamma, x, d)$ says to add an edge labelled γ connected to the vertex x in the new graph, for every β -labelled edge connecting a σ -labelled vertex to the non-terminal. d then indicates whether this rule applies to in-edges or out-edges of the non-terminal. For the formal definition of this substitution operation, see e.g. [17].

Definition 3 (edNCE Graph Grammar [17]). An *edNCE Graph Grammar* is a tuple $G = (\Sigma, \Delta, \Gamma, \Omega, P, S)$, where Σ is the alphabet of vertex labels, $\Delta \subseteq \Sigma$ is the alphabet of terminal vertex labels, Γ is the alphabet of edge labels, $\Omega \subseteq \Gamma$ is the alphabet of final edge labels, P is a finite set of productions and $S \in \Sigma - \Delta$ is the initial nonterminal. Productions are of the form $X \rightarrow (D, C)$, where $X \in \Sigma - \Delta$ is a non-terminal vertex and $(D, C) \in GRE_{\Sigma, \Gamma}$ is a graph with embedding.

A *derivation* in an edNCE grammar is a sequence of substitutions of non-terminals starting from the the graph which just contains the single starting non-terminal S . The set of all graphs (not containing non-terminals) isomorphic to some graph reachable in this manner is called the *language* of the grammar. edNCE grammars with the additional property that the order in which non-terminals are expanded is irrelevant are called *confluent* edNCE, or C-edNCE, grammars. We will focus on a special case:

Definition 4 (B-edNCE grammar). A *boundary edNCE*, or B-edNCE, grammar is a grammar such that for all productions $p : X \rightarrow (D, C)$, D contains no adjacent non-terminal nodes and C contains no connection instructions of the form $(\sigma, \beta/\gamma, x, d)$ where σ is a non-terminal label.

2.2 String graphs and rewriting

Definition 5 (String Graph). For disjoint sets $\mathcal{N} = \{N_f, N_g, \dots\}$, $\mathcal{W} = \{W_A, W_B, \dots\}$, a (directed) *string graph* is a directed graph labelled by the set $\mathcal{N} \cup \mathcal{W}$, where vertices with labels in \mathcal{N} are called *node-vertices* and vertices with labels in \mathcal{W} are called *wire-vertices*, and the following conditions hold: (1) there are no edges directly connecting two node-vertices (2) the in-degree of every wire-vertex is at most one and (3) the out-degree of every wire-vertex is at most one.

The category **SGraph** has as its objects string graphs and its morphisms string graph homomorphisms (i.e. graph homomorphisms respecting labels). In full generality, string graphs also allow one to restrict which node-vertices can be connected to which wire-vertices, and allow for an ordering on in- and out-edges (e.g. for non-commutative maps), but for simplicity, we will consider the case where any node-vertex can be connected to any wire-vertex and the ordering is irrelevant.

We will depict wire-vertices as small black dots and node-vertices as larger nodes of various shapes and colours. We can define undirected string graphs analogously, by replacing the last two conditions with the requirement that each wire-vertex have degree at most 2. To avoid excessive duplication we will state all of our results for the directed case, but similar results carry through to the undirected case. Thus, we will occasionally give undirected examples when they are more convenient than their directed counterparts.

Definition 6 (Inputs, Outputs and Boundary Vertices). A wire-vertex of a string graph G is called an *input* if it has no incoming edges. A wire-vertex with no outgoing edges is called an *output*. The boundary of G consists of all of its inputs and outputs.

Wires in string diagrams are geometric in nature. They are encoded in string graphs as chains of wire-vertices.

Definition 7. A *wire* is a maximal connected subgraph of a string graph consisting of only wire-vertices and at least one edge. There are three cases: (a) it forms a simple directed cycle, which is called a *circle*, (b) it is a chain where one or both endpoints are connected to node-vertices, which is called an *attached wire*, or (c) it is a chain not connected to any node-vertices, which is called a *bare wire*.

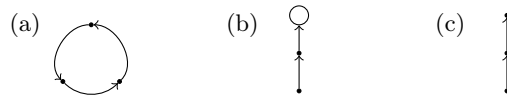


Fig. 4: Types of wires: (a) circle, (b) attached wire, (c) bare wire

In particular, when considering embeddings of string diagrams into each other, wires are allowed to be sub-divided arbitrarily. To accommodate this behaviour with string graphs, we represent wires as chains of wire-vertices. Of course, this choice is not unique, as we can represent a single wire with a chain of wire-vertices of any length. Being isomorphic up to the number of wire-vertices representing each wire is a natural notion of ‘sameness’ for string diagrams, which is called *wire-homeomorphism*.

Definition 8 (Wire-homeomorphic string graphs). Two string graphs G and G' are called *wire-homeomorphic*, written $G \sim G'$ if G' can be obtained from G by either merging two adjacent wire-vertices (left) or by splitting a wire-vertex into two adjacent wire-vertices (right) any number of times:

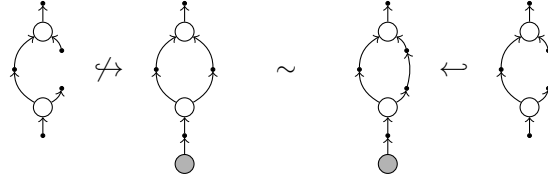
$$\dots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots \quad \mapsto \quad \dots \longrightarrow \bullet \longrightarrow \dots \qquad \dots \longrightarrow \bullet \longrightarrow \dots \quad \mapsto \quad \dots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots$$

Two string graphs $G \sim G'$ are "semantically" the same and only differ by the length of some of its wires. Note, that for any string graph G , its wire-homeomorphism class has a unique minimal representative, which can be obtained from G by just contracting wires as much as possible, so wire-homeomorphism is decidable.

In order to rewrite a string graph using a string graph rewrite rule, one first finds a matching of the LHS. Note we say an edge is incident to a subgraph $K \subseteq G$ if it connects a vertex in K to a vertex in $G \setminus K$.

Definition 9. Let L be a string graph with boundary $B \subseteq L$. Then a *matching* of L onto a string graph H is an injective string graph homomorphism $m : L \rightarrow \tilde{H}$ where $H \sim \tilde{H}$ and the only edges incident to the image $m(L) \subseteq \tilde{H}$ are also incident to $m(B)$.

In other words, a matching satisfies the ‘no dangling wires’ condition with respect to the boundary of L . Note that matching is done modulo wire-homeomorphism. This allows wires in the target graph to grow if necessary to injectively embed the pattern. In the example below, the embedding on the left fails, but the embedding into a wire-homeomorphic graph (right) succeeds:



As usual, string graph rewrite rules are encoded as spans $L \leftarrow B \rightarrow R$, where B is the common boundary of L and R .

Definition 10. A *rewrite* of a string graph G by a rule $L \leftarrow B \rightarrow R$ using a matching $m : L \rightarrow \tilde{G}$ (for $\tilde{G} \sim G$) consists of a pushout complement (1) followed by a pushout (2) in **SGraph**:

$$\begin{array}{ccccc} L & \longleftarrow & B & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{G} & \longleftarrow & I & \longrightarrow & H \end{array} \quad \begin{array}{c} (1) \\ \lrcorner \end{array} \quad \begin{array}{c} (2) \\ \lrcorner \end{array}$$

It was shown in [10] that for any matching of the LHS of a string graph rewrite rule, the pushout complement (1) and the pushout (2) exist and are unique, so DPO rewriting for string graph rewrite rules is well-defined.

3 Encoded string graphs and B-ESG grammars

We will introduce a type of context-free graph grammar suitable for defining families of string graphs, which is essentially a restriction of the class of B-edNCE grammars. However, to squeeze out a bit more expressive power, rather than using such grammars to generate string graphs themselves, we use them to generate *encoded string graphs*. An encoded string graph allows us to ‘fold’ some collection of fixed subgraphs into single edges, which will allow us more flexibility when it comes to the types of languages we can produce.

Definition 11 (Encoded string graph). Let $\mathcal{E} = \{\alpha, \beta, \dots\}$ be a finite set of *encoding symbols*. An *encoded string graph* is a string graph where we additionally allow edges labelled by encoding symbols $\alpha \in \mathcal{E}$ to connect pairs of node-vertices.

Definition 12 (Decoding system). A *decoding system* T is a set of DPO rewrite rules of the form:

$$\text{○} \xrightarrow{\alpha} \text{●} \quad \Rightarrow \quad \text{○} \rightarrow \dots \rightarrow \text{●}, \quad (3.1)$$

one for every triple $(\alpha, N_1, N_2) \in \mathcal{E} \times \mathcal{N} \times \mathcal{N}$, where the LHS consists of a single edge labeled α connecting an N_1 -labelled node-vertex to an N_2 -labelled node-vertex, and the RHS is a connected string graph that contains no inputs, outputs, or encoding labels.

Note, the invariant part of (3.1) consists of the two shared node-vertices. Thus, by construction T is confluent (since no two rules apply in the same location) and terminating (since no encoding labels occur in the RHS of a rule). Thus, *decoding* an encoded string graph consists of normalising with respect to T .

Throughout the rest of the paper, we assume that all of our grammars use the same vertex and edge label alphabets, Σ and Γ respectively, and also the same initial non-terminal S . The alphabet for terminal vertex labels is $\Delta := \mathcal{N} \cup \mathcal{W}$. We do not allow any non-final edge labels. We assume $\mathcal{E} \subset \Gamma$ and any edge with label in \mathcal{E} will be called an *encoding edge*.

Definition 13 (B-ESG grammar). A *B-ESG* grammar is a pair $B = (G, T)$, where T is a decoding system and $G = (\Sigma, \Delta, \Gamma, \Gamma, P, S)$ is a B-edNCE grammar, such that for every production $X \rightarrow (D, C) \in P$, the following conditions are satisfied:

- N1: An edge carries an encoding label if and only if it connects a pair of node-vertices.
- N2: Any connection instruction of the form $(N, \alpha/\beta, x, d)$ where N is a node-vertex label and x is a node-vertex, must have $\beta \in \mathcal{E}$.
- W1: Every wire-vertex in D has in-degree at most one and out-degree at most one.

- W2: There are no connection instructions of the form $(\sigma, \alpha/\beta, x, d)$ where σ is any vertex label and x is a wire-vertex.
- W3: For W a wire-vertex label and each γ and d , there is at most one connection instruction of the form $(W, \gamma/\delta, x, d)$ and we must have $\delta \notin \mathcal{E}$.
- W4: Let y be a non-terminal vertex with label Y in D . If y is adjacent to a wire-vertex labelled W via an edge with direction d and label β or there's a connection instruction of the form $(W, \alpha/\beta, y, d)$, then all productions Y must contain a connection instruction of the form $(W, \beta/\gamma, z, d)$.

The conditions N1 and N2 guarantee that node-vertices never become directly connected by an edge, unless that edge has an encoding label. W1-3 ensure that wires never ‘split’, i.e. wire-vertices always have at most one input or output. The final condition, which won't be necessary until the next section, ensures that inputs stay inputs and outputs stay outputs in a sentential form throughout the course of the derivation.

Definition 14 (B-ESG concrete derivation). A *concrete derivation* for a B-ESG grammar $B = (G, T)$ with S the initial non-terminal for G , consists of a derivation $S \Rightarrow_*^G H_1$ in G , where H_1 contains no non-terminals, followed by a decoding $H_1 \Rightarrow_*^T H_2$. We will denote such a concrete derivation as $S \Rightarrow_*^G H_1 \Rightarrow_*^T H_2$ or simply with $S \Rightarrow_*^B H_2$ if the graph H_1 is not relevant for the context.

Note that in the above definition S refers to both the initial non-terminal label, but also to a graph with a single vertex with label S which is the starting graph for a derivation. As usual, the *language* of a B-ESG grammar B is given by $L(B) := \{H \mid S \Rightarrow_*^B H\}$.

Theorem 1. Every graph in the language of a B-ESG grammar is a string graph.

Proof. Let $B = (G, T)$ be a B-ESG grammar. Decoding an encoded string graph will always produce a string graph, so it suffices to show that any derivation from G produces an encoded string graph. We call a sentential form an ESG-form if it is an encoded string graph, which possibly has some additional non-terminals that are either connected to node-vertices or are connected to wire-vertices in such a way that all wire-vertices have at most 1 in-edge and 1 out-edge. We show that any derivation starting from an ESG-form is an encoded string graph. This can be done by induction on the length of derivations. If the derivation is length 0, the sentential form has no non-terminals, so it is an encoded string graph. Otherwise, consider a derivation of length n . After the first step, any newly-introduced wire-vertices will have in-degree and out-degree at most 1 by W1 and W2, whereas the degrees of any already existing wire-vertices will not increase by W3. N1 and N2 will ensure that any resulting node-vertices will only be connected by edges with encoding labels, so the result is an ESG-form. Thus we can apply the induction hypothesis. Noting that S is, in particular, an ESG-form completes the proof. \square

Lemma 1. For every B-ESG grammar $B = (G, T)$, there exists $n \in \mathbb{N}$, such that if $H \in L(B)$ then H does not contain a wire with size bigger than n .

Proof. Let n be the length of the longest wire in the bodies of the productions in G and T , and consider an arbitrary sentential form obtained from $S \Rightarrow_*^G H'$. From condition $W2$, we see that expanding any of the non-terminals cannot create a new edge between an already established wire-vertex and a newly created wire-vertex. Therefore, a concrete derivation in G will produce an encoded string graph with maximum length of any wire n . Then, while doing the decoding, T will only replace edges between node-vertices and therefore a wire longer than n cannot be established. \square

Naturally, we want to be able to decide if a given string graph is in a B-ESG grammar. Since there should be no distinction between wire-homeomorphic string graphs, we state the membership problem as follows:

Problem 1 (Membership). Given a string graph H and a B-ESG grammar B , does there exist a string graph $\tilde{H} \sim H$, such that $\tilde{H} \in L(B)$? In such a case, construct a derivation sequence $S \Rightarrow_* \tilde{H}$.

Theorem 2. The membership problem for B-ESG grammars is decidable.

Proof. First, we show that exact membership (i.e. not up to wire-homeomorphism) is decidable. From Theorem 1, we know that any concrete B-ESG derivation produces an encoded string graph which is then decoded to a string graph. Since the decoding sequence can only increase the size of a graph, we can limit the problem to considering all graphs of size smaller than H . However, there are finitely many graphs whose size is smaller than H . For each such graph H' , we can then decide if $H' \in L(G)$ (this is the membership problem for B-edNCE grammars). Finally, we check if $H' \Rightarrow_*^T H$ which is also clearly decidable. If no such graph H' exists, then the answer is no and otherwise the answer is yes.

We now generalise to the wire-homeomorphic case. There may be infinitely many string graphs \tilde{H} such that $\tilde{H} \sim H$, but by using Lemma 1, it suffices to consider only those \tilde{H} which do not have wires longer than some fixed $n \in \mathbb{N}$. There are finitely many of these, so we can check if at least one of them is in $L(B)$ as before. Finally, since derivation sequences are recursively enumerable, if $\tilde{H} \in L(B)$, we can also construct a concrete derivation sequence $S \Rightarrow_* \tilde{H}$. \square

In section 5, we will show how to use B-ESG grammars for rewriting. To do this, we must show that the grammar produces some graphs which can be matched onto a given string graph, and ideally that the set of *all* matchings is finite, so that we can enumerate all of the relevant equalities. This is not true in general, but it is true whenever the B-ESG grammar satisfies some simple conditions:

Definition 15 (Match-exhaustive B-ESG grammar). We say that a B-ESG grammar $B = (G, T)$ is *match-exhaustive*, if (1) any production $X \rightarrow (D, C)$ which contains a bare wire in D has a finite bound on the number of times it can be expanded in any sentential form (2) no production contains an isolated wire-vertex in its body (3) there are no empty productions and (4) there are no productions consisting of a single node which is non-terminal.

Note that condition (1) can be easily decided by examining all productions which could possibly lead back to themselves via non-terminals, and seeing if they contain any bare wires. In [17] it was furthermore shown that any grammar can be transformed into an equivalent grammar satisfying conditions (3) and (4).

Problem 2 (Match-enumeration). Given a string graph H and a B-ESG grammar B , enumerate all of the B-ESG concrete derivations $S \Longrightarrow_*^B K$, such that there exists a matching $m : K \rightarrow \tilde{H}$ for some $\tilde{H} \sim H$.

Theorem 3. The match-enumeration problem for a B-ESG grammar B is decidable if B is a match-exhaustive grammar.

Proof. Let W be the number of wires in a string graph H . Then, for any $\tilde{H} \sim H$, we know that \tilde{H} must have the same number of wires and node-vertices as H . However, the number of wire-vertices in \tilde{H} may be arbitrarily large.

Condition (1) of Definition 15 implies that there exists $n \in \mathbb{N}$, such that, for any $K \in L(B)$, K has at most n bare wires. Any matching of string graphs will map at most two non-bare wires onto a single wire. So, if K has a matching on \tilde{H} , then it can have at most $2W$ non-bare wires. Therefore, K can have at most $2W + n$ wires. From Lemma 1 we know that the length of any such wire is bounded. Condition (2) of Definition 15 implies that any wire-vertex in K is part of some wire, thus there is a bound on the number of wire-vertices in K and we already know the number of node-vertices is also bounded. Thus, there are finitely many $K \in L(B)$ which could possibly have a matching onto some $\tilde{H} \sim H$, so we can enumerate them.

Finally, conditions (3) and (4) from Definition 15 imply that the sentential forms of B can only increase in size and therefore for any K satisfying the above conditions there are finitely many concrete derivations $S \Longrightarrow_*^B K$ which we can enumerate. \square

4 B-ESG rewrite patterns

In the previous section we introduced a new type of grammar which generates string graphs and which has some important decidability properties. Thus, we can use a single B-ESG grammar to represent a single family of string graphs. However, we still have not described how to encode equalities between families of string graphs. This is the primary contribution of this section. We introduce

the notions of *B-ESG rewrite pattern* and *B-ESG pattern instantiation* which show how this can be achieved in a formal way and then we give examples of important equalities which cannot be encoded using the !-graph formalism, but are expressible using B-ESG rewrite patterns.

Definition 16 (B-ESG rewrite pattern). A *B-ESG rewrite pattern* is a pair of B-ESG grammars $B_1 = (G_1, T)$ and $B_2 = (G_2, T)$, where $G_1 = (\Sigma, \Delta, \Gamma, \Gamma, P_1, S)$ and $G_2 = (\Sigma, \Delta, \Gamma, \Gamma, P_2, S)$, such that there is a bijective correspondence between the productions in P_1 and P_2 given by $X \rightarrow (D_1, C_1) \in P_1$ iff $X \rightarrow (D_2, C_2) \in P_2$ and the corresponding pairs of productions satisfy the following conditions:

- NT: There is a bijection between the non-terminal nodes in D_1 and the non-terminal nodes in D_2 which also preserves their labels.
- IO: There is a bijection between the inputs (resp. outputs) in D_1 and the inputs (resp. outputs) in D_2 .

Condition *NT* ensures that we can perform identical derivation sequences on both grammars G_1 and G_2 in the sense that we can apply the same order of productions to corresponding non-terminal vertices. This should become more clear from definition 17. When working with B-ESG rewrite patterns, without loss of generality, we will assume that the corresponding inputs/outputs/non-terminal nodes are identified between the two grammars by sharing the same name, instead of using a bijective function explicitly.

Definition 17 (B-ESG pattern instantiation). Given a B-ESG rewrite pattern (B_1, B_2) , a B-ESG pattern instantiation is given by a pair of concrete derivations:

$$S \xRightarrow{B_1}_{v_1, p_1} H_1 \xRightarrow{B_1}_{v_2, p_2} H_2 \xRightarrow{B_1}_{v_3, p_3} \dots \xRightarrow{B_1}_{v_n, p_n} H_n \xRightarrow{T}_* F$$

and

$$S \xRightarrow{B_2}_{v_1, p_1} H'_1 \xRightarrow{B_2}_{v_2, p_2} H'_2 \xRightarrow{B_2}_{v_3, p_3} \dots \xRightarrow{B_2}_{v_n, p_n} H'_n \xRightarrow{T}_* F'$$

In other words, we use an identical derivation sequence in the two B-edNCE grammars to get two encoded string graphs, which are then uniquely decoded using the productions of T . These ideas are similar to the *pair grammars* approach presented in [16], but different in that we are using a more general notion of grammar, our extension to the context-free grammars is more limited and our focus is on string diagram reasoning rather than computer program representation. These ideas have been further generalised in [19].

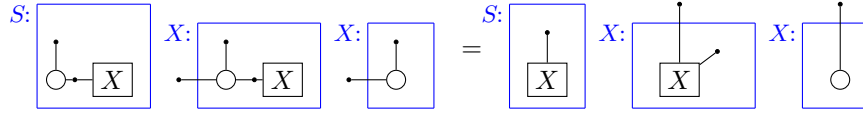
Theorem 4. Every B-ESG pattern instantiation is a string graph rewrite rule.

Proof. Consider a concrete derivation as in Definition 17. From Theorem 1, we know that F and F' are string graphs. We have to show that they have the same set of inputs/outputs. Note, that the decoding process using the productions

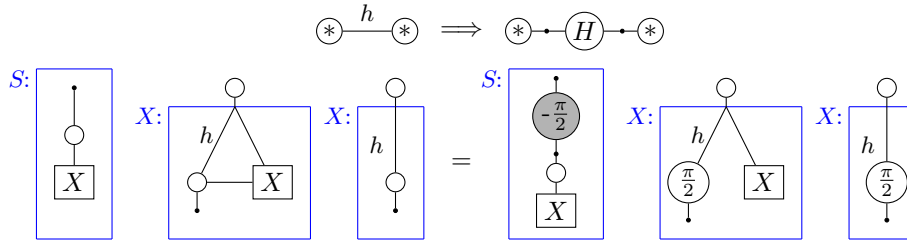
of T cannot establish any new inputs or outputs and thus we can reduce the problem to showing that H_n and H'_n have the same sets of inputs/outputs.

Setting $H_0 := S := H'_0$, we can prove by induction that any pair of sentential forms (H_i, H'_i) , has the same set of inputs. This is trivially true for (H_0, H'_0) . Assuming that (H_k, H'_k) have the same set of inputs, observe that (H_{k+1}, H'_{k+1}) is obtained by applying a corresponding pair of productions to (H_k, H'_k) . These productions satisfy all of the listed conditions in Definition 13 and Definition 16. In particular, conditions $W3$ and $W4$ guarantee that the in-degree of all wire-vertices in both H_k and H'_k won't be affected. Condition $W2$ implies that the newly created wire-vertices in H_{k+1} and H'_{k+1} are not connected to any of the previously established vertices in H_k and H'_k respectively. Combining this with condition IO ensures that any newly created wire-vertices are inputs in H_{k+1} iff they are inputs in H'_{k+1} . Thus, H_{k+1} and H'_{k+1} have the same set of inputs. In particular, H_n and H'_n have the same inputs. By symmetry, H_n and H'_n have the same output vertices. \square

As mentioned in the introduction, we now gain previously non-existent expressive power for families of string graph rewrite rules. For instance, we can write a rule that merges a chain of white node-vertices, each with 1 input, into a single white node-vertex with n inputs as follows:



In the introduction, we also gave an example of a rule involving clique-like graphs. In fact, a minor modification to (1.4) yields a rule that is directly relevant for quantum computation. The following rule, called *local complementation*:



is crucial to establishing the completeness of a rewrite system called the *ZX-calculus*, which has been used extensively for reasoning about quantum circuits and measurement-based quantum computation. This rule uses a non-trivial decoding system, where the h -labelled edges are interpreted as wires containing a single node-vertex. Here we are using ‘ $*$ ’ to mean ‘any node-vertex label’. The node-vertices labelled $\pm \frac{\pi}{2}$ represent *quantum phase gates*, whereas the H is a *Hadamard gate*. The interested reader can find a detailed description of the ZX-calculus in e.g. [7] the completeness theorem based on local complementation in [2].

5 Transforming B-ESG grammars

In the previous section, we showed how a B-ESG rewrite pattern can encode (infinitely) many equalities between string diagrams, where we represent the equalities as string graph rewrite rules. Thus, we can use B-ESG rewrite patterns to encode axioms and axiom schemas of a string diagram theory. However, in order to perform equational reasoning between families of string diagrams, we need to show how we can use such axioms in our formalism in order to obtain new equalities. This is the primary contribution of this section. In particular, we show how we can transform a B-ESG rewrite pattern into another one using a string graph rewrite rule in an admissible way. Finally, we generalise this result by showing how we can transform B-ESG rewrite patterns using another B-ESG rewrite pattern in an admissible way.

Definition 18 (Final subgraph). Given a production $X \rightarrow (D, C)$ in a B-edNCE grammar, we say that a subgraph S of D is final if S is a string graph which is not adjacent to any non-terminals in D and no vertex of S has associated connection instructions in C .

Definition 19 (B-ESG transformation step). We say that a given grammar $B = (G, T)$ can be transformed into another grammar $B' = (G', T)$ using a given string graph rewrite rule $SR = L \leftarrow I \rightarrow R$, if (1) there exists a production $p = X \rightarrow (D, C)$ in G , such that L can be matched into a final subgraph of D (2) G' has the same productions as G , except for p which has been modified to $p' := (D', C)$ where D' is the result of applying the rewriting rule SR to the matching of (1).

Theorem 5. Given a B-ESG grammar B , the pair (B, B') is a B-ESG rewrite pattern, where B' is the result of a B-ESG transformation step applied to B .

Proof. Let $B' = (G', T)$. Assume that the modified production is $p = X \rightarrow (D, C)$ of $B = (G, T)$ and the corresponding production in G' is $p' = X \rightarrow (D', C)$. Rewriting $D \rightsquigarrow D'$ only affects a final subgraph S of D and thus it cannot establish any edges between non-terminals or modify the connection instructions. Therefore, G' is a B-edNCE grammar and conditions $N2, W2, W3, W4$ and NT are satisfied. Conditions $W1$ and $N1$ are also preserved, because we are replacing a string graph S with another string graph and the edges between vertices in $D \setminus S$ and S are preserved. Thus, B' is a B-ESG grammar. Finally, condition IO is satisfied, because string graph rewriting does not create or remove inputs/outputs. \square

Theorem 6. Let B be a B-ESG grammar, let SR be a string graph rewrite rule and let (B, B') be the B-ESG rewrite pattern induced by B and SR as per Theorem 5. Then, (B, B') is admissible in the sense that any pattern instantiation $S \xRightarrow{B} F$ together with $S \xRightarrow{B'} F'$ are such that the string graph F' can be obtained from F via repeated applications of SR .

Proof. Let p be the production of B which is modified by SR and let p' be the result of the modification. Because the modification is done on a final subgraph S of p , as p is expanded, a copy of S is created which is not adjacent to any previously established vertices. Moreover, the copy S is not adjacent to any non-terminals and thus its neighbourhood won't change in the following sentential forms. Therefore, to obtain F' from F , we have to apply the rule SR exactly n times, where n is the number of times the production p appears in the pattern instantiation. \square

Corollary 1. If B is a B-ESG grammar and (B_1, B_2) is a B-ESG rewrite pattern with B_1 match-exhaustive, then we can enumerate the pattern instantiations of (B_1, B_2) which induce an admissible rewrite pattern (B, B') .

Proof. From Theorem 4, any pattern instantiation of (B_1, B_2) is a string graph rewrite rule. If such a string graph rewrite rule matches a final subgraph in some production of B , then it induces an admissible rewrite pattern (B, B') as shown in Theorem 6. Finally, by Theorem 3, we can enumerate all of those instantiations and thus all such B' . \square

6 Conclusion and future work

In this paper, we have defined a family of grammars that is suitable for describing languages of string graphs and string graph rewrite rules. We have also showed that these languages admit nice decidability properties. There are two natural directions in which to extend this work. Firstly, the conditions of a B-ESG grammar are sufficient, but not necessary to obtain a language consisting of only string graphs. One might ask if there exist other natural and easily-decidable conditions yielding such languages, and how those conditions relate to the B-ESG conditions.

The second, and more compelling direction for future work is the development of tools for reasoning with B-ESG rewrite patterns and, most importantly, deriving new patterns. We took the first step in this direction in Section 5, where we described a fairly limited technique whereby grammars can be transformed by using string graph rewrite rules to rewrite the final parts of some productions. However, in the case string graph patterns based on !-boxes, which we briefly discussed in the introduction, we additionally have the ability to do genuine pattern-on-pattern rewriting, and derive new !-box rules using !-box induction. We expect both of these techniques to extend to the context-free case. Thus, we would ultimately like a much richer notion of rewriting, whereby B-ESG patterns can be used to rewrite non-final parts of grammars, where e.g. non-terminals are allowed to match on non-terminals, subject to suitable consistency conditions. Similarly, we intend to extend !-box induction to a more general structural induction principle that can be used to derive B-ESG rewrite patterns from basic string graph rules. This will provide a method for deriving infinite

families of rules which is both extremely powerful and capable of producing machine-checkable diagrammatic proofs.

Acknowledgements. We would like to thank the anonymous reviewers for their feedback. We also gratefully acknowledge financial support from EPSRC, the Scatcherd European Scholarship, and the John Templeton Foundation.

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